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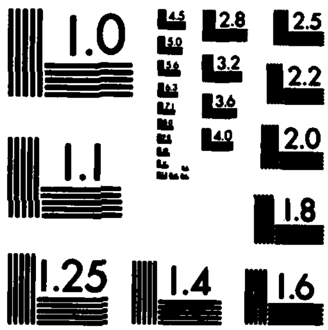

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A TANDEM STORAGE SYSTEM AND ITS DIFFUSION LIMIT

by

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A Tandem Storage System and Its Diffusion Limit

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Abstract. We consider a two-dimensional diffusion process $Z(t) = [Z_1(t), Z_2(t)]$ that lives in the half strip $\{0 \leq Z_1 \leq 1, 0 \leq Z_2 < \infty\}$. On the interior of this state space, Z behaves like a standard Brownian motion (indpendent components with zero drift and unit variance), and there is instantaneous reflection at the boundary. The reflection is in a direction normal to the boundary at $Z_1 = 1$ and $Z_2 = 0$, but at $Z_1 = 0$ the reflection is at an angle θ below the normal $(0 < \theta < \pi/2)$. This process Z is shown to arise as the diffusion limit of a certain tandem storage or queueing system. It is shown that $Z(t)$ has a non-defective limit distribution F as $t \to \infty$, and the marginal distributions of F are computed explicitly. The marginal limit distribution for Z_1 is uniform (this result is essentially trivial), but that for Z_2 is much more complicated.

Key Words and Phrases. Tandem Queues, Storage Theory, Diffusion Processes, Wiener-Hopf Technique, Reflected Brownian Motion.

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1. Introduction and Summary

This paper is concerned with approximate analysis of a certain discrete-time storage process. The physical system giving rise to this two-dimensional process is pictured in Figure 1 below. It consists of a buffer (or dam or reservoir) having finite capacity b, followed by another buffer having infinite capacity. We denote by S_t^1 and S_t^2 the contents of the finite buffer and the infinite buffer respectively at time $t = 0, 1, \ldots$. Imposing very particular assumptions on the input process to the first buffer, the transfer process between buffers, and the output process from the second buffer, we study here the approximate behavior of the storage process $S_t = (S_t^1, S_t^2)$ as $b \uparrow \infty$. To be more specific, let us define a continuous-time process $S^*(t) = [S_1^*(t), S_2^*(t)]$ by setting

$$(1.1) \qquad S_1^*(t) \equiv \frac{1}{b} S^1_{[b^2t]} \quad \text{and} \quad S_2^*(t) \equiv \frac{1}{\sigma b} S^2_{[b^2t]}$$

for $t \geq 0$, where $[x]$ denotes the integer part of x (the largest integer less than or equal to x). There is a two-dimensional diffusion process Z such that S^* converges weakly to Z as $b \to \infty$, and we would like to determine the steady-state behavior of Z.

At the boundary, Z reflects instantaneously. On the sides $Z_1 = 1$ and $Z_2 = 0$ we have what is called normal reflection. On the side $Z_1 = 0$ there is oblique reflection in the direction pictured in Figure 2, where $a \equiv 1/\sigma$ and σ^2 is the variance parameter introduced in Figure 1. The precise meaning of this boundary behavior will be explained in §3. In §4 it will be shown that $P\{Z(t) \leq y\} \to F(y)$ as $t \to \infty$, where F is a non-defective distribtion on the strip Σ, and this limit distribution (or steady-state distribution) will be shown to satisfy a

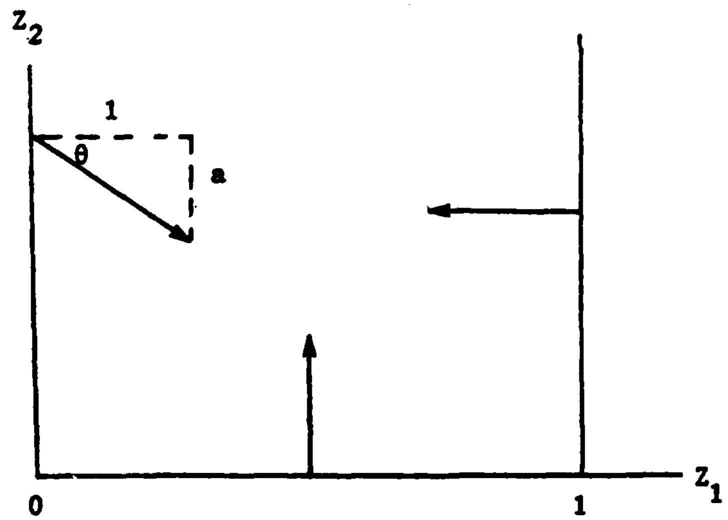

Figure 2. State Space Σ and Angles of Reflection for the Diffusion Limit Z

certain adjoint relation. Using this adjoint relation and Wiener-Hopf techniques, we calculate the marginal distributions of F in §5.

The marginal steady-state distribution of Z_1 is uniform, which follows immediately from the fact that Z_1 is a one-dimensional standard

Brownian motion constrained to $[0,1]$ by reflecting barriers. But the marginal steady-state distribution of Z_2,

$$G(\xi) \equiv F(1,\xi) , \qquad u \geq 0 ,$$

is much more complex. Let

$$c \equiv \pi\, \Gamma(1 - \tfrac{\theta}{\pi})/\Gamma(1 - \tfrac{2\theta}{\pi})\, \Gamma(\tfrac{\theta}{\pi}) , \tag{1.2}$$

$$p(u) \equiv e^{-\theta u}(1 - e^{-\pi u})^{-2\theta/\pi} , \qquad u \geq 0 , \tag{1.3}$$

where $\theta \equiv \tan^{-1} a$ as in Figure 2, and $\Gamma(\cdot)$ is the standard gamma function

$$\Gamma(s) \equiv \int_0^\infty e^{-t}\, t^{s-1}\, dt , \qquad s > 0 .$$

It will be shown that $cp(u)$ is a probability density function on $(0,\infty)$ with mean $1/a$, and that this distribution describes the asymptotic behavior of a certain increasing process associated with the boundary $Z_1 = 0$. Our main result is the following.

(1.4) <u>Theorem</u>. $G(d\xi) = g(\xi)d\xi$ where $g(\xi) = a\int_\xi^\infty cp(u)du$.

From (1.4) it follows that the steady-state marginal distribution G has Laplace transform transfer between the two buffers. As a convenient normalization we can assume that $Var(u_t^0) = 1$, and then we set $\sigma^2 \equiv Var(u_t^2)$. These notational conventions are also displayed in Figure 1. For simplicity we take $S_0^1 = S_0^2 = 0$, meaning that both buffers are initially empty.

The content process for the finite buffer is now defined inductively by

$$(2.3) \qquad S_t^1 = \begin{cases} 0 , & \text{if } \ 0 \geq S_{t-1}^1 + u_t^0 - u_t^1 \\ S_{t-1}^1 + u_t^0 - u_t^1 , & \text{if } \ 0 \leq S_{t-1}^1 + u_t^0 - u_t^1 \leq b \\ b , & \text{if } \ S_{t-1}^1 + u_t^0 - u_t^1 \geq b \end{cases}$$

for $t = 1, 2, \ldots$. This is of course the standard construction for the content process of a finite dam in discrete time [6]. Actual input to the dam during period t may be less than the full potential input u_t^0 because of the capacity restriction, and actual output may similarly be less than the full potential output u_t^1 because of the restriction $S_2^1 \geq 0$. Defining

$$(2.4) \qquad x_t^1 \equiv u_t^0 - u_t^1 \quad ,$$

$$(2.5) \qquad y_t^0 \equiv [S_{t-1}^1 + x_t^1 - b]^+ \quad ,$$

and

$$(2.6) \qquad y_t^1 \equiv [S_{t-1}^1 + x_t^1]^- \ ,$$

we observe that y_t^0 represents lost potential input to the finite buffer during period t, and y_t^1 represents lost potential output from the finite buffer, which amounts to lost potential transfer between the two buffers buffer, which amounts to lost potential transfer between the two buffers pictured in Figure 1. That is, actual input to the finite buffer during period t is $u_t^0 - y_t^0$, actual transfer between the two buffers during period t is $u_t^1 - y_t^1$, and (2.3) is equivalent to (2.4)-(2.6) plus

$$(2.7) \qquad S_t^1 \equiv S_{t-1}^1 + x_t^1 + y_t^1 - y_t^0 \quad ,$$

for $t = 1, 2, \ldots$. Having established that the actual input to the infinite buffer during period t is $u_t^1 - y_t^1$, we can define the content process for the infinite buffer inductively by setting

$$(2.8) \qquad S_t^2 \equiv \begin{cases} S_{t-1}^2 + u_t^1 - y_t^1 - u_t^2 & \text{if} \quad S_{t-1}^2 + u_t^1 - y_t^1 - u_t^2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $t = 1, 2, \ldots$. Defining

$$(2.9) \qquad x_t^2 \equiv u_t^1 - u_t^2 \; ,$$

$$(2.10) \qquad y_t^2 \equiv [S_{t-1}^2 + x_t^2 - y_t^1]^- ,$$

we see that y_t^2 represents lost potential output from the infinite buffer during period t. That is, actual output during period t is $u_t^2 - y_t^2$, and (2.8) is equivalent to (2.9)-(2.10) plus

$$(2.11) \qquad S_t^2 = S_{t-1}^2 + x_t^2 - y_t^1 + y_t^2 \quad , \qquad \text{for } t = 1, 2, \ldots .$$

To derive an equivalent and very useful characterization of the storage process S, let us define the cumulative quantities

$$(2.12) \qquad X_t^k = x_1^k + \cdots + x_t^k \ , \qquad \text{for} \quad t = 0, 1, \ldots \quad \text{and} \quad k = 1, 2,$$

$$(2.13) \qquad Y_t^k = y_1^k + \cdots + y_t^k \ , \qquad \text{for} \quad t = 1, 2, \ldots \text{ and } \quad k = 0, 1, 2,$$

with $X_0^k = Y_0^k = 0$. By summing (2.7) and (2.11) over periods $1, \ldots, t$ we have

$$(2.14) \qquad S_t^1 = X_t^1 + Y_t^1 - Y_t^0 \quad ,$$

$$(2.15) \qquad S_t^2 = X_t^2 + Y_t^2 - Y_t^1$$

for $t = 0, 1, \ldots$. By construction we have

$$(2.16) \qquad 0 \leq S_t^1 \leq b \quad \text{and} \quad S_t^2 \geq 0 \qquad \text{for} \quad t = 0, 1, \ldots$$

and one can easily verify that

$$(2.17) \qquad S_t^1 \Delta Y_t^1 = (b - S_t^1)\ \Delta Y_t^0 = S_t^2 \Delta Y_t^2 = 0 \qquad \text{for} \quad t = 1, 2, \ldots,$$

where $\Delta Y_t^k \equiv Y_t^k - Y_{t-1}^k = y_t^k$. Obviously (2.16) and (2.17) together require that $\Delta Y_t^1 = 0$ except when $S_t^1 = 0$, that $\Delta Y_t^0 = 0$ except when $S_t^1 = b$, and that $\Delta Y_t^2 = 0$ except when $S_t^2 = 0$.

We now define the normalized continuous-time storage process $S^*(t) = [S_1^*(t), S_2^*]$ by applying to S the transformation (1.1). In a similar vein, let $X^*(t) = [X_1^*(t), X_2^*(t)]$ and $Y^*(t) = [Y_0^*(t), Y_1^*(t), Y_2^*(t)]$ be defined by

$$(2.18) \qquad X_1^*(t) = \frac{1}{b} X^1_{[b^2 t]} \ , \qquad X_2^* = \frac{1}{\sigma b} X^2_{[b^2 t]} \ ,$$

$$(2.19) \qquad Y_0^*(t) = \frac{1}{b}\, Y^0_{[b^2t]} \ , \ Y_1^*(t) = \frac{1}{b}\, Y^1_{[b^2t]} , \ Y_2^*(t) = \frac{1}{\sigma b}\, Y^2_{[b^2t]}$$

for $t \geq 0$. Obviously X^* is right-continuous with left-hand limits (RCLL), and from (2.7), (2.11), and (2.14)-(2.17) it follows that Y^* and S^* have the following properties:

$$(2.20) \quad Y_k^* \text{ is RCLL and non-decreasing with } Y_k^*(0) = 0 \qquad (k = 0, 1, 2),$$

$$(2.21) \quad S_1^*(t) = X_1^*(t) + Y_1^*(t) - Y_0^*(t) \quad ,$$

$$(2.22) \quad S_2^*(t) = X_2^*(t) + Y_2^*(t) - aY_1^*(t) \qquad \text{where } a \equiv \frac{1}{\sigma} ,$$

$$(2.23) \quad 0 \leq S_1^*(t) \leq 1 \qquad \text{and} \qquad S_2^*(t) \geq 0 \ ,$$

$$(2.24) \quad \int_0^t S_1^*(u)\ dY_1^*(u) = \int_0^t [1-S_1^*(u)]\ dY_0^*(u) = \int_0^t S_2^*(u)\ dY_2^*(u) = 0$$

for all $t \geq 0$, where the integrals in (2.24) are defined path-by-path in the Lebesgue-Stieltjes sense. (These integrals do not exist in the Riemann-Stieltjes sense because, for example, S_1^* and Y_1^* jump simultaneously.)

We have arrived at (2.20)-(2.24) as characterizations of processes constructed previously, but these relationships can actually be used to <u>define</u> Y^* and S^* in terms of X^*. That is, with X^* defined in terms of the primitive sequences $\{u_t^k\}$ via (2.4), (2.9), (2.12) and (2.18), there exists a <u>unique</u> pair of processes (Y^*,S^*) that jointly satisfy (2.20)-(2.24). Thus (2.20)-(2.24) implicitly define an operator which maps X^* into (Y^*,S^*). In the next section we use relationships analagous to (2.20)-(2.24) to define a pair of processes (U,Z) in terms of a two-dimensional standard Brownian motion W. Donsker's Theorem [2] tells us that X^* converges weakly to W as $b \to \infty$, and then the continuous mapping theorem [2] can be used to show that

$$(X^*, Y^*, S^*) \Rightarrow (W, U, Z) \qquad \text{as} \quad b \to \infty ,$$

where $\Rightarrow$ denotes weak convergence in a seven-fold product space. We shall not go further into the convergence argument here, referring the interested reader to Wenocur [7]. The remainder of this paper is devoted to construction and characterization of the diffusion limit Z.

3. The Limiting Diffusion Process

Let $W(t) = [W_1(t), W_2(t)]$ be a two-dimensional standard Brownian motion (independent components with zero shift and unit variance). Defining $a \equiv 1/\sigma$ as in (2.22), we wish to construct processes $U(t) = [U_0(t), U_1(t), U_2(t)]$ and $Z(t) = [Z_1(t), Z_2(t)]$ which jointly satisfy

(3.1) U_k is continuous and non-decreasing with $U_k(0) = 0$ $(k = 0,1,2,)$,

(3.2) $Z_1(t) = W_1(t) + U_1(t) - U_0(t)$,

(3.3) $Z_2(t) = W_2(t) + U_2(t) - aU_1(t)$,

(3.4) $0 \le Z_1(t) \le 1$ and $Z_2(t) \ge 0$,

$$(3.5)\qquad \int_0^t Z_1(s)\, dU_1(s) = \int_0^t [1-Z_1(s)]\, dU_0(s) = \int_0^t Z_2(s)\, dU_2(s) = 0$$

for $t \geq 0$. Condition (3.4) says that Z lives within the semi-infinite strip Σ pictured in Figure 2. Obviously (3.5) requires that U_1 increase only when $Z_1 = 0$, that U_0 increase only when $Z_1 = 1$, and that U_2 increase only when $Z_2 = 0$. Conditions (3.2) and (3.3) determine the direction in which Z is driven upon hitting the boundary of Σ. On each of the three boundary surfaces we have reflection in the (constant) direction pictured in Figure 2 above.

For each (continuous) sample path W there is in fact a unique pair of sample paths (U,Z) satisfying (3.1)-(3.5). The argument goes as follows. If we delete all mention of U_2 and Z_2 in (3.1)-(3.5), the remaining conditions are those which define Z_1 as a one-dimensional standard Brownian motion restricted to $[0,1]$ by reflecting barriers; U_1 and U_0 are increasing processes (local time processes) associated with the lower boundary and upper boundary respectively. For a construction of U_0 and U_1 in terms of W_1 see §5 of [5]. With U_0 and U_1 determined, one may argue as in §2 of [3] that the remaining requirements of (3.1)-(3.5) are uniquely satisfied by taking

$$(3.6)\qquad U_2(t) = \sup_{0 \leq s \leq t} [-W_2(s) + aU_1(s)]^+ , \qquad t \geq 0 .$$

This construction of (U,Z) from W is valid for any staring state $W(0) = (x_1,x_2) \in \Sigma$. Holding $\{W(t) - W(0), t \geq 0\}$ fixed, it can be verified that

(3.7) $Z_1(t)$ and $Z_2(t)$ are non-decreasing functions of both x_1 and x_2 for all $t \geq 0$.

Using an argument like that in §2 of [3], it follows that Z is a Markov process, and we shall denote by $P_x(\cdot)$ the distribution on the path space of Z corresponding to initial state $Z(0) = W(0) = x$. From (3.6) we have that

(3.8) $F_t(x,y) \equiv P_x\{Z(t) \leq y\}$ is non-increasing in both x_1 and x_2 for all $t \geq 0$ and $y \in \Sigma$.

The process U satisfying (3.1)-(3.5) is non-anticipating with respect to W. Thus Z is a continuous semimartingale, using the filtration generated by W, and we can develop its analytical theory using the multidimensional Ito Formula. For functions $f : R^2 \to R$ that are twice continuously differentiable, we define the differential operators

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} ,$$

$$D_0 \equiv -\frac{\partial}{\partial x_1} , \; D_1 \equiv \frac{\partial}{\partial x_1} - a\frac{\partial}{\partial x_2} \quad \text{and} \quad D_2 \equiv \frac{\partial}{\partial x_2} .$$

Defining the boundary surfaces

$$\Sigma_0 \equiv \{(x_1,x_2) \in \Sigma : x_1 = 1\} ,$$

$$\Sigma_1 \equiv \{(x_1,x_2) \in \Sigma : x_1 = 0\} ,$$

and

$$\Sigma_2 \equiv \{(x_1,x_2) \in \Sigma : x_2 = 0\} ,$$

we observe that $D_k f$ is the directional derivitive of f in the direction of reflection associated with Σ_k (see Figure 2). The following proposition is virtually identical to Theorem 2 of [3], so we shall not prove it.

(3.9) <u>Proposition</u>. If $f : R^2 \to R$ is twice continuously differentiable, then for all λ and $t \geq 0$

$$e^{\lambda t} f(Z(t)) = f(Z(0)) + \sum_{k=1}^{2} \int_0^t e^{\lambda s} \frac{\partial}{\partial x_k} f(Z(s))\, dW_k(s)$$

$$+ \int_0^t e^{\lambda s}(\lambda + \tfrac{1}{2}\Delta) f(Z(s))\, ds$$

$$+ \sum_{k=0}^{2} \int_0^t e^{\lambda s} D_k f(Z(s))\, dU_k(s) .$$

Here the integrals involving dW_k are of Ito type, while those involving dU_k are defined path by path in the Riemann-Stieltjes sense.

(3.10) <u>Proposition</u>. $E_x[U_0(t)] \sim \frac{1}{2} t$, $E_x[U_1(t)] \sim \frac{1}{2} t$, and $E_x[U_2(t)] \sim \frac{a}{2} t$ as $t \to \infty$ $(x \in \Sigma)$.

<u>Proof</u>. Set $\lambda = 0$ and $f(x_1,x_2) = x_1$ in (3.9). Then $(\lambda+\Delta)f = 0$, $-D_0 f = D_1 f = 1$, and $D_2 f = 0$. Moreover, the Ito integrals have zero expectation, because their integrands are bounded. Thus, taking E_x of both sides, we obtain

(3.11) $$E_x[Z_1(t)] = x_1 + E_x[U_1(t)] - E_x[U_0(t)] .$$

Similarly, by taking $\lambda = 0$ and $f(x) = x_2$ we have

Taking E_x of both sides in (3.16), dividing by t and letting $t \to \infty$ gives (3.14), thus completing the proof.

(3.17) Proposition. Let $\lambda > 0$, $\alpha \in (0, \pi/2)$ and $\beta > 0$ be constants, let $f(x) = \cos(\alpha(1-x_1)) \exp(\beta x_2)$, and define $M(t) = \exp(\lambda t) f(Z(t))$, $t \geq 0$. If the constants λ, α, β simultaneously satisfy

(3.18) $$\lambda + \frac{1}{2}(\beta^2 - \alpha^2) = 0 ,$$

(3.19) $$\alpha \sin \alpha - a\beta \cos \alpha = 0 ,$$

then the stopped process $\{M(t \wedge T), t \geq 0\}$ is a supermartingale, where

(3.20) $$T = \inf\{t \geq 0 : Z_2(t) = 0\} .$$

Remark. The stopped process is in fact a martingale, but we shall only need the weaker property, and it is slightly easier to prove.

Proof. First observe that

(3.21) $$D_0 f(1, x_2) = 0 ,$$

regardless of how λ, α and β are chosen, and that (3.18) and (3.19) imply

(3.22) $$(\lambda + \frac{1}{2}\Delta) f(x_1, x_2) = 0 ,$$

and

(3.23) $$D_1 f(0, x_2) = 0$$

respectively. Combining (3.21)-(3.23) with the fact that $U_2(T) = 0$, we see that all terms on the right side of (3.9) except for the Ito integrals are zero, provided $t \leq T$. Thus $\{M(t \wedge T)\}$ is a local martingale. But f is a positive function, hence M is a positive process, and a positive local martingale is a supermartingale (Fatou's Lemma). This completes the proof.

(3.24) Propositon. $E_x(T) < \infty$ for all $x \in \Sigma$, where (3.20) defines T.

Proof. First we show that there exist constants $\lambda > 0$, $\alpha \in (0, \pi/2)$ and $\beta > 0$ satisfying (3.18) and (3.19). Obviously (3.19) is equivalent to

$$\beta = \frac{\alpha}{a} \tan \alpha \quad . \tag{3.25}$$

Substituting this into (3.19), we arrive at the requirement that

$$\phi(\alpha) \equiv \alpha^2(1 - a^{-2} \tan^2 \alpha) = 2\lambda \; . \tag{3.26}$$

It can be verified that $\phi(0) = 0$, $\phi'(0) > 0$, and $\phi(\alpha) \to -\infty$ as $\alpha \uparrow \pi/2$, so there exists $\alpha \in (0, \pi/2)$ satisfying (3.26) if $\lambda > 0$ is chosen small enough. Choosing β according to (3.25), we then have a triple (λ, α, β) satisfying the hypotheses of Proposition (3.17). Thus $M(t \wedge T)$ is a supermartingale, meaning that $E_x[M(0)] \geq E[M(t \wedge T)]$, which is equivalent to

$$f(x) \geq E_x[e^{\lambda(t \wedge T)} f(Z(t \wedge T))] \geq E_x[e^{\lambda(t \wedge T)}] \cos \alpha \; . \tag{3.27}$$

Letting $t \uparrow \infty$ in (3.27) gives

(3.28) $$E_x(e^{\lambda T}) \leq f(x)/\cos\alpha$$

which of course implies $E_x(T) < \infty$ as desired.

4. The Analytical Problem

Hereafter it will be assumed that $W(0) = Z(0) = 0$, and the symbols $P(\cdot)$ and $E(\cdot)$ will be understood to mean $P_0(\cdot)$ and $E_0(\cdot)$ respectively. Similarly, recalling the notation $F_t(x,y) \equiv P_x\{Z(t) \leq y\}$, let us agree to write

(4.1) $$F_t(y) \equiv P\{Z(t) \leq y\} = F_t(0,y) \ .$$

Because the origin is the least element of our strip Σ (under the usual partial ordering), it follows from the monotonicity property (3.8) that $Z(t)$ is stochastically increasing in t, meaning that $F_t(y)$ is a non-increasing function of t for each $y \in \Sigma$. Consequently,

(4.2) $$F(y) \equiv \lim_{t \to \infty} F_t(y)$$

exists, and it follows from Proposition (3.24) that F must furthermore be a non-defective distribution function. Our objective is to compute the marginal distribution

(4.3) $$G(\xi) \equiv F(1,\xi) = \lim_{t \to \infty} P\{Z_2(t) \leq \xi\} \ , \qquad \xi \geq 0 \ .$$

Incidentally, there is an elegant coupling argument which shows that $F_t(x,y) \to F(y)$ as $t \to \infty$ for <u>all</u> starting states $x \in \Sigma$, but we shall have no need for this fact. (The key observation is that if two sample paths of W have different starting states but identical increments, then the corresponding paths of Z will interesect at some finite time T and be identical thereafter.)

We now want to derive an analytical characterization of the steady-state distribution F. A useful first step is to define the functions

$$(4.4) \qquad H_t(u) = \frac{2}{t} E\left[\int_0^t 1_{\{Z_2(s) \le u\}} \, dU_1(s)\right] , \qquad t \ge 0 .$$

Obviously, $H_t(\cdot)$ is increasing with $H_t(\infty) = 2E[U_1(t)]/t$, which one can easily show to be finite. Proposition (3.10) shows that $H_t(\infty) \to 1$ as $t \to \infty$.

(4.5) Proposition. $H_t \Rightarrow H$ as $t \to \infty$, where H is a distribution function on $[0,\infty)$ with

$$(4.6) \qquad \int_0^\infty u \, H(du) = \int_0^\infty [1 - H(u)] \, du = \frac{1}{a} .$$

Furthermore,

$$(4.7) \qquad G(\xi) = a \int_0^\xi [1 - H(u)] du .$$

Proof. It will be useful to define the distribution functions

$$(4.8) \qquad G_t(\xi) \equiv P\{Z_2(t) \le \xi\} = F_t(1,\xi) , \qquad \xi \ge 0 .$$

Let $\alpha > 0$ be arbitrary, put into (3.9) the test function $f(x_1,x_2) = \exp(-\alpha x_2)$, and take $E(\cdot)$ of both sides. The Ito integrals have zero expectation (integrands are bounded), $D_0 f = 0$, $D_1 f = a\alpha f$, $D_2 f(x_1,0) = -\alpha$, and $\Delta f = \alpha^2 f$, so we have

$$(4.9) \qquad E[e^{-\alpha Z_2(t)}] = 1 + \frac{1}{2}\alpha^2 E[\int_0^t e^{-\alpha Z_2(s)}\, ds]$$

$$+ a\alpha E[\int_0^t e^{-\alpha Z_2(s)}\, dU_1(s)] - \alpha E[U_2(t)] \ .$$

Set $G_t^*(\alpha) \equiv \int_{[0,\infty]} e^{-\alpha\xi}\, G_t(d\xi)$, and let other Laplace transforms be denoted similarly. Since $G_t \Rightarrow G$, the continuity theorem gives

$$(4.10) \qquad \frac{1}{t} E[\int_0^t e^{-\alpha Z_2(s)}\, ds] = \frac{1}{t}\int_0^t E[e^{-\alpha Z_2(s)}]\, ds$$

$$= \frac{1}{t}\int_0^t G_s^*(\alpha)\, ds \to G^*(\alpha) \qquad \text{as } t \to \infty \ .$$

Of course

$$(4.11) \qquad \frac{1}{t} E[U_2(t)] \to \frac{1}{2} a \qquad \text{as } t \to \infty$$

by (3.10), and a monotone class argument gives

$$(4.12) \qquad \frac{1}{t} E[\int_0^t e^{-\alpha Z_2(s)}\, dU_1(s)] = \frac{1}{2}\int_{[0,\infty)} e^{-\alpha\xi}\, H_t(d\xi) \equiv \frac{1}{2} H_t^*(\alpha) \ .$$

Dividing (4.9) through by t, letting $t \to \infty$, and using (4.10)-(4.13), we obtain $H_t^*(\alpha) \to 1 - \alpha\, G^*(\alpha)/a$ as $t \to \infty$. Thus $H_t \Longrightarrow H$, where H is an increasing function whose Laplace transform is related to that of G via

(4.13) $$G^*(\alpha) = \frac{a}{\alpha}[1 - H^*(\alpha)] \ .$$

Since G is a distribution function it follows from (4.13) that H is also a distribution function with mean given by (4.6). Finally, (4.13) is the transform version of (4.7), so the proof is complete.

For the next proposition, we want to insert in the Ito Formula (3.9) test functions $f : \Sigma \to R$ such that

(4.14) f is twice continuously differentiable, with bounded first and second-order partials on Σ, and $D_0 f(1,x_2) = D_2 f(x_1,0) = 0$.

(4.15) Proposition. If f satisfies (4.14) then

(4.16) $$0 = \int_\Sigma \Delta f(x)\, F(dx) + \int_{[0,\infty)} D_1 f(0,u)\, H(du) \ .$$

Proof. Set $\lambda = 0$ in (3.9) and take $E(\cdot)$ of both sides. The Ito integrals have zero expectation (bounded integrands), and $D_0 f(1,x_2) = D_2 f(x_1,0) = 0$ by hypothesis, so we obtain

Since ν_k concentrates all of its mass on Σ_k by (3.5), we can rewrite (4.18) as

$$(4.20) \qquad 0 = \int_S \frac{1}{2} \Delta f g(x)\ F(dx) + \sum_{k=0}^{2} \int_{\Sigma_k} D_k f(\sigma)\ \nu_k(d\sigma) \quad ,$$

and (3.10) shows tht $\nu_0(\Sigma_0) = \nu_1(\Sigma_1) = 1/2$ and $\nu_2(\Sigma_2) = a/2$. Obviously, (4.15) specializes (4.20) to test functions f of the class (4.14), with $H(u) \equiv 2\nu_1[0,u]$. For more on adjoint relations like (4.20) see §9 of [4].

5. The Steady-State Distribution

By putting a well chosen test function f into the adjoint relation (4.15), we can determine the steady-state boundary distribution H defined by (4.4)-(4.5). Let

$$(5.1) \qquad \phi^+(z) \equiv \int_{[0,\infty)} e^{izu}\ H(du) \ , \qquad \mathrm{Im}(z) \geq 0 \ ,$$

$$(5.1) \qquad \phi^-(z) \equiv \int_{[0,\infty)} e^{-izu} H(du) \ , \qquad \mathrm{Im}(z) \geq 0 \ ,$$

(5.3) <u>Proposition</u>. ϕ^+ and ϕ^- are bounded, analytic in the upper half-plane and lower-half plane respectively, and satisfy

(5.4) $$\frac{\phi^{-}(z)}{\phi^{+}(z)} = \frac{a \cosh z - i \sinh z}{a \cosh z + i \sinh z} \qquad \text{for } z \text{ real .}$$

<u>Proof</u>. The boundedness and analyticity follow immediately from the definitions (5.1), (5.2). To prove (5.4), let $z \in R$ be arbitrary and define

$$f(x) \cosh izx_2 \cosh z(1-x_1) \ , \qquad \text{for } x \in \Sigma \ .$$

It is easy to verify that f satisfies all the restrictions of (4.14), plus $\Delta f = 0$, so (4.15) gives us

(5.5) $$0 = \int_{[0,\tau)} D_1 f(0,u) \ H(du) \quad .$$

(This is actually a pair of identities, one for the real part of $D_1 f$ and one for the imaginary part, but we shall continue to use the efficient notation of complex variables.) Now direct computation gives

(5.6) $$D_1 f(0,u) \equiv \left(\frac{\partial}{\partial x_1} - a \frac{\partial}{\partial x_2}\right) f(0,u)$$

$$= - z \sinh z \cosh izu - aiz \cosh z \sinh izu$$

$$= - \frac{1}{2} z\left[(\sinh z + ai \cosh z) \ e^{izu} + (\sinh z - ai \cosh z) \ e^{-izu}\right]$$

$$= \frac{1}{2} iz\left[(a \cosh z + i \sinh z) \ e^{-izu} - (a \cosh z - i \sinh z) \ e^{izu}\right] \ .$$

Substituting (5.6) into (5.5) and dividing in the obvious way gives the desired identity (5.4).

Using the identity $\Gamma(\alpha)\,\Gamma(1-\alpha) = \pi/\sin(\pi\alpha)$, we divide (5.11) by (5.10) to obtain

$$\frac{\psi^-(z)}{\psi^+(z)} = \frac{\sin(\theta - iz)}{\sin(\theta + iz)} \quad . \tag{5.12}$$

Expanding the sines on the right side of (5.12), we find that (5.4) holds with (ψ^+,ψ^-) in place of (ϕ^+,ϕ^-). Furthermore, p is positive and integrable on $(0,\infty)$, so (ψ^+,ψ^-) have <u>all</u> the properties ascribed to (ϕ^+,ϕ^-) in Proposition (5.3).

(5.13) <u>Proposition</u>. $\phi^+(z) = c\psi^+(z)$, where c is defined by (1.2). Thus $H(du) = cp(u)du$.

<u>Proof</u>. Let $k(z) = \phi^+(z)/\psi^+(z)$ for $\text{Im}(z) \geq 0$, $k(z) = \phi^-(z)/\psi^-(z)$ for $\text{Im}(z) \leq 0$. These two definitions agree for real z because (ϕ^+,ϕ^-) and (ψ^+,ψ^-) both satisfy (5.4). Recall that the Γ function has no zeros, and its singularities are poles on the negative real axis, cf. 6.1.3 of Abramowitz and Stegun [1]. From this and the analyticity of ϕ^+, ϕ^-, ψ^+, ψ^- on their respective half planes, it follows that $k(z)$ is an entire function of z. Further,

$$\Gamma(z+a)\, z^{b-a}/\Gamma(z+b) \to 1 \tag{5.14}$$

as $|z| \to \infty$, $z \neq -a-n$ or $-b-n$ $(n = 0, 1, \ldots)$, cf. 6.1.4 of Abramowitz and Stegun [1]. From (5.10) and (5.11) we then have

$$|\psi^+(z)| \geq \gamma |z|^{1-2\theta/\pi} \quad \text{as } |z| \to \infty,\ \text{Im}(z) \geq 0 , \tag{5.15}$$

(5.16) $$|\phi^-(z)| \geq \gamma |z|^{1-2\theta/\pi} \quad \text{as} \quad |z| \to \infty,\ \mathrm{Im}(z) \leq 0 ,$$

for some constant $\gamma > 0$. Since ϕ^+ and ϕ^- are bounded in their respective half-planes, it follows that

(5.17) $$|k(z)| \leq A|z|^{-(1-2\theta/\pi)} \quad \text{as} \quad |z| \to \infty ,$$

for some constant $A > 0$. Since $0 < 1-2\theta/\pi < 1$, it follows from (5.17) and Liouville's Theorem that such an entire function $k(z)$ is a constant. So $\phi^+(z) = c\psi^+(z)$ for some $c > 0$. Of course $\phi^+(0) = 1$ by definition (ϕ^+ is the characteristic function of a probability distribution), so it must be that $c = 1/\psi^+(0)$, so (5.10) shows that c is given by (1.2) as claimed. This proves the first statement of the proposition, and the second follows from the continuity theorem for characteristic functions.

The preceeding argument gives no hint of the means originally used to calculate p, which was roughly the following. First the right side of (5.4) was factored into products of zeros, using the Hadamard product formulas for the numerator and denominator individually. The individual factors were allocated to ψ^+ and ψ^- so as to insure analyticity in the upper half-plane and lower half-plane respectively, and then the Euler product for Γ was used to obtain (5.10) and (5.11). Finally, a partial fraction decomposition and sum revealed that ψ^+ is the characteristic function of the density p defined by (1.3).

To obtain the final formula for G stated earlier as Theorem (1.4), one simply combines (5.13) and (4.7). The equivalent Laplace transform solution (1.5) is obtained by combining (4.13), (5.13) and (5.10).

and it jumps upon hitting the boundary. If the boundary is hit at a point $(x,1)$, then there is an instantaneous jump to the point $(x-a\varepsilon \operatorname{sgn} x,\ 1-\varepsilon)$, after which the Brownian movement resumes. If the boundary is hit at $(x,-1)$, then the jump is to $(x-a\varepsilon \operatorname{sgn} x,\ -1+\varepsilon)$. Let T_n be the nth time at which the boundary is hit, and set $X_n \equiv X(T_n)$. We claim that $\{X_n\}$ is a Markov chain, that it has a unique stationary distribution, and that the stationary distribution of $|X_n|$ converges weakly to the boundary distribution H of §4 as $\varepsilon \downarrow 0$. We shall make no attempt to justify any of these claims, hoping that the symmetry apparent in Figure 3 makes them at least plausible.

Hereafter assume that the initial state of (X,Y) is distributed randomly over the boundary of the strip according to the stationary distribution of $\{X_n\}$. Then the transition mechanism for our stationary Markov chain $\{X_n\}$ can be expressed as

(6.1) $$X_{n+1} = X_n - a\varepsilon \operatorname{sgn} X_n + \xi_{n+1} ,$$

where ξ_{n+1} is the increment of Brownian movement (in the X direction) between consecutive hittings of the boundary. Defining

(6.2) $$\phi_\varepsilon(z) \equiv E[\exp(izX_n)] ,$$

$$(6.3) \qquad \phi_\varepsilon(z) \equiv E[\exp(iz\xi_{n+1}] ,$$

we have from (6.1) that

$$(6.4) \qquad \phi_\varepsilon(z) = \phi_\varepsilon(z)\, E\{\exp[iz(X_n - a\varepsilon \text{ sgn } X_n)]\} ,$$

because $\{X_n\}$ is stationary and ξ_{n+1} is independent of X_n (using the spatial homogeneity of Brownian motion). Next, using the fact that

$$M(t) \equiv \exp\{iz\, X(t) + zY(t)\} , \qquad t \geq 0 ,$$

is a martingale for arbitrary complex z, one can easily derive the formula

$$(6.5) \qquad \phi_\varepsilon(z) = \cosh(z(1-\varepsilon))/\cosh z .$$

Now defining

$$(6.6) \qquad \phi_\varepsilon^+(z) = E[\exp(iz|X_n|)] ,$$

$$(6.7) \qquad \phi_\varepsilon^-(z) = E[\exp(iz|X_n|)] ,$$

symmetry and (6.2) give us

$$(6.8) \qquad \phi_\varepsilon(z) = \frac{1}{2}\phi_\varepsilon^+(z) + \frac{1}{2}\phi_\varepsilon^-(z)$$

$$(6.9) \qquad E\{\exp[iz(X_n - a\varepsilon \text{ sgn } X_n)]\}$$

$$= \frac{1}{2}\phi_\varepsilon^+(z)\, e^{-iza\varepsilon} + \frac{1}{2}\phi_\varepsilon^-(z)\, e^{iza\varepsilon} .$$

Substituting (6.5), (6.8) and (6.9) into (6.4) and simplifying, we obtain

$$(6.10) \qquad -\frac{\phi_\varepsilon^-(z)}{\phi_\varepsilon^+(z)} = \frac{\cosh z - \exp(-iza\varepsilon)\cosh(z(1-\varepsilon))}{\cosh z - \exp(iza\varepsilon)\cosh(z(1-\varepsilon))} \quad .$$

Let $\phi^+(z)$ and $\phi^-(z)$ be defined in terms of H as in §5. If the stationary distribution of $|X_n|$ converges weakly to H as $\varepsilon \downarrow 0$, as claimed earlier, then of course $\phi_\varepsilon^+(z) \to \phi_\varepsilon^+(z)$ and $\phi_\varepsilon^-(z) \to \phi^-(z)$ as $\varepsilon \downarrow 0$. Assuming the weak convergence, we may then let $\varepsilon \downarrow 0$ in (6.10) to obtain (5.4).

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ABSTRACT - Report Number 204

We consider a two-dimensional diffusion process $Z(t) = [Z_1(t), Z_2(t)]$ that lives in the half strip $\{0 \leq Z_1 \leq 1, 0 \leq Z_2 < \infty\}$. On the interior of this state space, Z behaves like a standard Brownian motion (indpendent components with zero drift and unit variance), and there is instantaneous reflection at the boundary. The reflection is in a direction normal to the boundary at $Z_1 = 1$ and $Z_2 = 0$, but at $Z_1 = 0$ the reflection is at an angle θ below the normal $(0 < \theta < \pi/2)$. This process Z is shown to arise as the diffusion limit of a certain tandem storage or queueing system. It is shown that $Z(t)$ has a non-defective limit distribution F as $t \to \infty$, and the marginal distributions of F are computed explicitly. The marginal limit distribution for Z_1 is uniform (this result is essentially trivial), but that for Z_2 is much more complicated.